

Galactic Halos are Einstein Clusters of WIMPs

Kayll Lake[*]

Department of Physics, Queen's University, Kingston, Ontario, Canada, K7L 3N6

(Dated: July 25, 2006)

It is shown that an Einstein cluster of WIMPs, WIMPs on stable circular geodesic orbits generating the static spherically symmetric gravitational field of a galactic halo, can exactly reproduce the rotation curve of any galaxy simply by adjusting the local angular momentum distribution and consequent number distribution of the WIMPs. No new physics is involved (assuming the forthcoming discovery of WIMPs). Further, stability of the orbits can require an inner truncation of the halo and an explicit example of this is given. There is no exact Newtonian counterpart to the model presented since pure Newtonian gravity fails to account for the contribution made by the angular momentum distribution of the particles in creating the gravitational field. In effect, a galactic halo is supported by the hoop stresses created by the orbiting WIMPs.

Many years ago Einstein [1] showed that one can introduce rotation without global angular momentum. He used a cluster of randomly distributed gravitating dust (non-colliding) particles each on a circular geodesic orbit about the center of symmetry of the gravitational field produced by the entire distribution of particles. In this letter I show that this model, in a modern context, can solve all dynamical issues associated with galactic rotation curves, one of the foremost problems in all of physical science today. It is proposed that the dark matter halos are the gravitational fields produced by massive particles (WIMPs) in orbit about galactic centers [2]. The gravitational contribution of the visible components is, in excellent approximation, simply ignored here.

Einstein's heuristic arguments can be made more precise [3] and here we simply sketch the construction of the energy-momentum tensor. Consider a static spherically symmetric continuous distribution of particles (WIMPs) of (rest) mass $m \neq 0$ and 4 - momenta p_α which are on stable circular geodesic orbits about the center of symmetry. (The existence and stability of these orbits is discussed below.) In terms of "curvature" coordinates the template for our calculations takes the form [4]

$$ds^2 = \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2, \quad (1)$$

where $d\Omega^2$ is the metric of a unit sphere ($d\theta^2 + \sin^2 \theta d\phi^2$). Now let n be the local number density of particles and at some spatial point \mathcal{P} (labelled, say, by $(r_{\mathcal{P}}, \theta_{\mathcal{P}}, \phi_{\mathcal{P}})$) construct

$$T_\alpha^\beta|_{\mathcal{P}} \equiv \frac{n}{m} \langle p_\alpha p^\beta \rangle \quad (2)$$

where the average $\langle \rangle$ is taken at \mathcal{P} and over all particles passing through \mathcal{P} . The total angular momentum L (defined as usual by $L^2 = p_\theta^2 + p_\phi^2 / \sin^2 \theta \neq 0$) is of course constant along each trajectory but we also assume that it is constant along every trajectory through \mathcal{P} thus arriving at the main results [1]

$$\langle p_\theta \rangle = \langle p_\phi \rangle = \langle p_\theta p_\phi \rangle = 0 \quad (3)$$

and

$$\langle p_\theta^2 \rangle = \frac{\langle p_\phi^2 \rangle}{\sin^2 \theta_{\mathcal{P}}} = \frac{L^2}{2}. \quad (4)$$

Moving to a continuous distribution and maintaining spherical symmetry we have $L = L(r)$ and $n = n(r)$ so that finally

$$T_t^t = -mn(r)(1 + \mathcal{L}^2(r)) \quad (5)$$

and

$$T_\theta^\theta = T_\phi^\phi = \frac{mn(r)}{2} \mathcal{L}^2(r) \quad (6)$$

where

$$\mathcal{L}(r) \equiv \frac{L(r)}{mr}. \quad (7)$$

All other components of T_α^β vanish and $\nabla_\beta T_\alpha^\beta$ vanishes identically.

Let us now make use of the standard Einstein equations $G_\alpha^\beta = 8\pi T_\alpha^\beta$ in the coordinates defined by (1) and using the energy-momentum tensor derived above. First, since $T_r^r = 0$, we can solve for $M(r)$ algebraically from $G_r^r = 0$ to obtain

$$M(r) = \frac{r^2 \Phi'}{1 + 2r\Phi'} \quad (8)$$

where $' \equiv d/dr$. Next, writing $-T_t^t/T_\theta^\theta = -G_t^t/G_\theta^\theta$ we arrive at

$$r\Phi' = \frac{\mathcal{L}^2(r)}{1 + \mathcal{L}^2(r)} \quad (9)$$

so that (8) can be written in the equivalent form

$$M(r) = \frac{r\mathcal{L}^2(r)}{1 + 3\mathcal{L}^2(r)}. \quad (10)$$

Finally, from (5) and (6) since $T_t^t + 2T_\theta^\theta = -mn$, we have

$$4\pi mn = \frac{(1 - r\Phi')(r\Phi'' + 2\Phi' + 2r(\Phi')^2)}{r(1 + 2r\Phi')^2}, \quad (11)$$

which, with the aide of (9), can also be given in the form

$$4\pi mn = \frac{\mathcal{L}(2r\mathcal{L}' + \mathcal{L}(1 + 3\mathcal{L}^2))}{r^2(1 + \mathcal{L}^2)(1 + 3\mathcal{L}^2)^2}. \quad (12)$$

Of course, mn has to be ≥ 0 . In this model there is no way to determine m and n separately (an advantage discussed below). We refer to the product nm as the “number distribution”. As (12) makes clear, in this model the number distribution mn can be considered a consequence of the angular momentum distribution \mathcal{L} .

If we were given $\mathcal{L}(r)$ then the gravitational field of a halo would be complete up to quadrature in (9). As discussed in detail previously [5], it is $\Phi(r)$ (and not $\mathcal{L}(r)$) which follows directly from observations of galactic rotation curves. However, important constraints can be placed on $\mathcal{L}(r)$. In the notation of [5], the existence of these circular orbits requires $0 < r\Phi' < 1$ which with (9) translates simply to $0 < \mathcal{L}(r)^2 < \infty$ but shows that the first term in (11) is positive. The stability of the circular geodesic orbits requires $3\Phi' + r\Phi'' > 2r(\Phi')^2$ which gives $r\mathcal{L}\mathcal{L}' > -\mathcal{L}^2$ and shows that the second term in (11) is positive only for $r\Phi' > 1/4$ ($\mathcal{L}^2 > 1/3$). A mapping onto the observer’s plane requires $\Phi' > r\Phi'' + 2r(\Phi')^2$ so that $r\mathcal{L}\mathcal{L}' < \mathcal{L}^2$ and so the distribution of angular momentum should satisfy

$$-\mathcal{L}^2 < r\mathcal{L}\mathcal{L}' < \mathcal{L}^2 \quad (13)$$

which, when applied to (12) directly repeats the important result: The stability of the circular geodesics orbits requires

$$\mathcal{L}^2 > \frac{1}{3}. \quad (14)$$

Let r_0 signify the border of stable orbits defined by $\mathcal{L}^2(r_0) = 1/3$. If $\mathcal{L}^2(r)$ is monotone increasing then the region $r < r_0$ must be excluded and the halo distribution is necessarily a thick shell. An explicit example of this is shown below. (If one was to measure the “relativistic” nature of the configurations by way of (say) $2M(r)/r$ then they are always highly “relativistic”, starting with $r \sim 6M(r)$ and becoming more “relativistic” ($r \sim 3M(r)$) as $\mathcal{L}^2 \rightarrow \infty$. This notion of “relativistic” is, however, based on the Schwarzschild vacuum ($M = \text{constant}$) and it is inappropriate here. This is discussed further below.)

As discussed previously [5], observations of galactic rotation curves are reported by way of the “optical convention” $v \equiv (\lambda_o - \lambda_e)/\lambda_e$ so that

$$\frac{\Phi' b^2}{r(1 - r\Phi')} = (v(b) - v(b=0))^2 \quad (15)$$

where the mapping from the observer’s plane to r is given by way of the classical impact parameter

$$b^2 = \frac{r^2}{e^{2\Phi}}. \quad (16)$$

Observations therefore give Φ up to quadrature and as a consequence the full geometry, the angular momentum distribution \mathcal{L} and consequently the number distribution mn . In any particular case one can take the view that the angular momentum distribution of the orbiting WIMPs has been adjusted to exactly match the observed rotation curve. Equivalently, the observed rotation curve is a consequence of the distribution \mathcal{L} .

With the understanding that the procedure used here can be applied to any rotation curve, we consider, as previously [5], the specific example of a universal curve given by [6]

$$v(b)^2 = \frac{\alpha b^2}{b^2 + \beta}, \quad (17)$$

where α and β are positive constants characteristic of a particular galaxy. With (17) it follows from (15) that

$$e^{2\Phi} = \frac{r^2}{\alpha r^2 \mathcal{W}(y) - \beta} \quad (18)$$

where \mathcal{W} is the Lambert W function [7],

$$y = \frac{c e^{\frac{\beta}{\alpha r^2}}}{\alpha^2 r^2}, \quad (19)$$

and c is a constant > 0 . Observe that with (18) and (19)

$$r\Phi' = \frac{1}{1 + \mathcal{W}}, \quad (20)$$

and so from (9) the required angular momentum distribution is given simply by

$$\mathcal{L}^2 = \frac{1}{\mathcal{W}}. \quad (21)$$

The gravitational mass follows from (8) and is given by

$$M = \frac{r}{3 + \mathcal{W}}. \quad (22)$$

Finally, from (11) or (12) the number distribution follows as

$$4\pi nm = \frac{\mathcal{W}(\alpha r^2(3 + 6\mathcal{W} + \mathcal{W}^2) + 2\beta\mathcal{W})}{(3 + \mathcal{W})^2(1 + \mathcal{W})^2 r^4 \alpha}. \quad (23)$$

In this model, mapping onto the observer’s plane requires $c > \alpha\beta$. With (21), limits on \mathcal{W} follow from the stability condition (14) and the existence condition for the orbits so that

$$0 < \mathcal{W} < 3. \quad (24)$$

Since $\mathcal{W}(y)$ is a monotone decreasing function of r , decreasing from ∞ at $r = 0$ to 0 as $r \rightarrow \infty$, condition (24) tells us that for stability the distribution is necessarily a thick “shell” with the central divergence in $\mathcal{W}(y)$ removed as mentioned above. The junction conditions associated with this inner truncation are discussed in detail

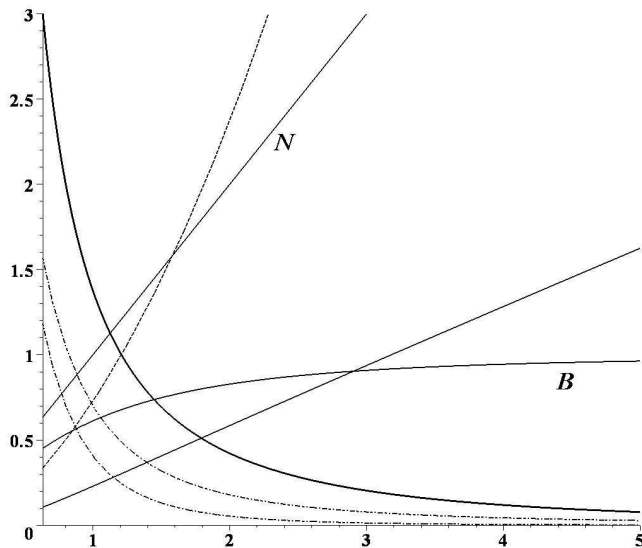


FIG. 1: Some representative properties of the model which follows from (17). For this figure $\alpha = \beta = 1$ and $c = 2$. These values are for demonstration purposes only. The thick solid curve gives $\mathcal{W}(y)$, and the thin solid curve gives $M(r)$ which is essentially linear. The dashed increasing curve gives \mathcal{L}^2 and the lower dashdot decreasing curve gives $8\pi mn(r)$ which lies below the effective density, the dashdot decreasing curve given by $8\pi mn(r)(1 + \mathcal{L}^2)$ (in these two cases 8π is included as an amplification factor for the image). The distribution is truncated when the orbits become unstable at $\mathcal{W} = 3$, in this case for $r \approx 0.633$. The curves labelled N and B are discussed below.

below. Some representative properties of this particular model are shown in Figure 1

Because of the stability condition (14), a discussion of junction conditions is appropriate. Geometrically, the junction is examined by way of the Darmois-Israel conditions - the continuity of the first and second fundamental forms at a boundary surface Σ [8]. For example, the smooth junction of metrics of the form (1), at any boundary surface defined by constant r , only requires the continuity of M and Φ' , assuming the continuity of r , θ and ϕ . As a result, G_r^r , but not G_θ^θ nor G_t^t , is necessarily continuous at r_Σ . As a consequence, since any Einstein cluster has $G_r^r = 0$, the matching conditions become trivial. One could, for example, insert an interior or exterior Schwarzschild field at any r_Σ simply by choosing the Schwarzschild mass M to be

$$M = \frac{r^2 \Phi'}{1 + 2r\Phi'}|_\Sigma. \quad (25)$$

This “Swiss-cheese” type of matching is, however, clearly inappropriate for the objects considered here [9]. Here the distribution of “dust” particles is supported by the hoop stresses given by (6) and junctions onto other forms of “dust” without these stresses are more appropriate in the present context. Consider, as a simple example, junction onto a uniform dust section (in particular, a

piece of a flat Robertson-Walker spacetime with density $\rho(t)$ [10] but not necessarily the background cosmology). Let the junction surface Σ be defined by $M/r = \epsilon$ where ϵ is a (dimensionless) constant. The continuity of the areal radii and effective gravitational masses then gives

$$r_\Sigma = a(t)r_\Sigma = c\sqrt{\frac{\epsilon}{C}}a(t)^{3/2} \quad (26)$$

where c is the velocity of light and C is the constant defined by $4\pi\rho(t)a(t)^3/3$. Unlike the “Swiss-cheese” type of model, here both r_Σ and r_Σ evolve with t .

It is interesting to compare the model presented here with what would result from pure Newtonian gravity. To do this consider the potential $\tilde{\Phi}$ and density $\tilde{\rho}$ defined by [11]

$$\nabla^2 \tilde{\Phi} = 4\pi\tilde{\rho} \equiv -R_t^t \quad (27)$$

where R_t^t is the time component of the Ricci tensor of (1). A direct computation shows that

$$\tilde{\rho} = mn \left(\frac{1 + r\Phi'}{1 - r\Phi'} \right) \quad (28)$$

where mn is given by (11), or equivalently from (9),

$$\tilde{\rho} = mn(1 + 2\mathcal{L}^2). \quad (29)$$

In pure Newtonian gravity we have

$$\rho = mn \quad (30)$$

and so pure Newtonian gravity fails to account for the contribution made by the angular momentum distribution of the orbiting particles. This contribution is in fact the sole support for the Einstein cluster.

We end with a few words about gravitational lensing. Of course any model of a galactic halo should not only reproduce the observed rotation curves, but also agree with any observed lensing. At present the possibility of such simultaneous measurements seems remote [12]. Here we only demonstrate that an Einstein cluster can be expected to be a significant source of gravitational lensing. The curves B and N in Figure 1 show the potential impact parameters \mathcal{B} [13] for non-radial null geodesics in the Newtonian case (N , no bending) and for the Einstein cluster described above (B , for the values of α, β and c stated). The flattening of B is clear from (22).

In summary, we have presented a model, based on an argument originally due to Einstein [1], that exactly reproduces the rotation curve of any galaxy. The model requires no new physics but does require the existence of WIMPs on stable circular geodesic orbits. It is these orbiting particles that create the gravitational field of the dark matter halo. The halo is held together by the hoop stress generated by the angular momentum of the particles. The actual value of the rest mass for these

particles plays no role in the argument which, given that this is not presently known, can be considered an advantage. There is no exact Newtonian counterpart to the model since pure Newtonian gravity fails to account for the contribution made by the angular momentum distribution of the particles in creating the gravitational field. This contribution is in effect the entire content of the model. The model predicts, on the basis of the stability of orbits, that the halo is shell-like and so the WIMPs do not occupy the inner regions, at least on stable circular orbits. The gravitational lensing properties of the model need to be compared with observations when these be-

come available. These properties involve an analysis of the junction of the halos onto both interior and exterior dust fields which are not supported by hoop stresses.

Acknowledgments

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. Portions of this work were made possible by use of *GRTensorII* [14].

[*] Electronic Address: lake@astro.queensu.ca

- [1] A. Einstein, Ann. Math. **40**, 922 (1939). Whereas Einstein was trying to show that the Schwarzschild horizon can not exist since it does not exist in this very reasonable model, this should not detract from the ingenious argument he presented.
- [2] The existence of dark matter is an integral part of the model discussed here. In this regard see D. Garfinkle, Class. Quant. Grav. **23**, 1391 (2006) [arXiv:\(gr-qc/0511082\)](#) and references therein.
- [3] See G. L. Comer, Gen. Rel. Grav. **28**, 601 (1996), G. L. Comer, D. Langlois and P. Peter, Class. Quantum Grav. **10**, L127 (1993), G. L. Comer and J. Katz, Class. Quantum Grav. **10**, 1751 (1993) and A. B. Evans, Gen. Rel. Grav. **8**, 155 (1977). This latter reference includes extensions beyond the static case that we do not use here. See also, G. Magli, Class. Quantum Grav. **14**, 1937 (1997), A. F. Teixeira and M. M. Som, J. Phys. A **7**, 838 (1974), H. Bondi, Gen. Rel. Grav. **2**, 321 (1971), B. K. Datta, Gen. Rel. Grav. **1**, 19 (1970), C. Gilbert, Mon. Not. R. Astr. Soc. **114**, 628 (1954).
- [4] We use geometrical units throughout, a signature of +2 and the summation convention. Functional dependence is shown usually only on the first appearance of a function.
- [5] K. Lake, Phys. Rev. Lett. **92**, 051101 (2004) [arXiv:\(gr-qc/0302067\)](#). Refer to the references given there and especially U. Nucamendi, M. Salgado and D. Sudarsky, Phys. Rev. D **63**, 125016 (2001) [arXiv:\(gr-qc/0011049\)](#) on extracting information from rotation curves. For a more recent discussion, including the possible use of gravitational lensing, see T. Faber and M. Visser [arXiv:\(astro-ph/0512213\)](#).
- [6] See M. Persic, P. Salucci and F. Stel, Mon. Not. R. Astr. Soc. **281**, 27 (1996) [arXiv:\(astro-ph/9506004\)](#). On the basis of a large number rotation curves, they have suggested a universal rotation curve intrinsic to galactic halos as given in (17). We continue to use it here as a demonstration to show how observations determine \mathcal{L} and therefore mn .
- [7] \mathcal{W} is defined by the condition $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$ and the function has wide application in the physical sciences. De-

spite the somewhat ugly nature of $\mathcal{W}(y)$, the view taken here is that it is simply an “elementary function”. See R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Advances in Computational Mathematics **5**, 329 (1996).

- [8] See, for example P. Musgrave and K. Lake, Class. Quantum Grav. **13**, 1885 (1996). [arXiv:\(gr-qc/9510052\)](#)
- [9] It is interesting to note, however, that on the small scale such a cluster around an object would make it appear more relativistic on the outer boundary.
- [10] We use r as the comoving coordinate in the uniform dust and $a(t)$ as the scale factor where t is the proper time for comoving observers.
- [11] See, for example, L. D. Landau and E. M. Lifchitz, *The Classical Theory of Fields*, (Pergamon Press 1975) or S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley and Sons, New York, 1972) or C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman, San Francisco, 1973).
- [12] See, for example Faber and Visser [5].
- [13] From (1) it follows that the evolution of non-radial null geodesics is governed by

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{r^2} \left(1 - \frac{2M(r)}{r}\right) \left(\frac{\mathcal{B}(r)^2}{b^2} - 1\right) \quad (31)$$

where λ is an affine parameter, b is a constant of motion (the classical impact parameter at infinity in the asymptotically flat case) and the potential impact parameter \mathcal{B} is given by

$$\mathcal{B}(r)^2 = \frac{r^2}{e^{2\Phi(r)}}. \quad (32)$$

The straight line motion of pure Newtonian gravity is given by $\mathcal{B}(r) = r$.

- [14] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address <http://grtensor.org>